DYNAMIC IMAGERY IN CHILDREN'S REPRESENTATIONS OF NUMBER

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An exploratory study of 92 high ability 3rd through 6th Graders was conducted to investigate links between their understanding of the structure of numeration and their representations of the counting sequence 1-100. From children's explanations and drawings of the numbers from 1-100, we seek to infer their internal imagistic representations. Our observations indicate that children showing evidence of dynamic internal representations and access to a variety of internal images have more developed relational understanding.

Research groups focussing on the conceptual development of numeration (Bednarz and Janvier, 1988; Cobb and Wheatley, 1988; Denvir and Brown, 1986; Fuson, 1990; Hiebert and Wearne, 1992; Kamii, 1989; Ross, 1990; Rubin and Russell, 1992) have highlighted children's construction of the number system. Children's representations of number as some form of physical, pictorial, or notational recording have been exemplified in many studies analysing children's structural development of number and conceptual understanding of numeration (Davis, Maher and Noddings, 1990; Goldin and Herscovics, 1991; Hiebert and Wearne, 1992; Hughes, 1986; Rubin and Russell, 1992; Thomas, 1992). Thomas (1992) reported the wide variety of mental pictures of the number sequence 1 to 100 that were used by children in a study of 40 children from Grades K through 4. It was found that although some aspects of structure appeared in the imagery of Grade 2 children most Grade 4 children still did not possess the structural flexibility with number to successfully mentally manipulate 2-digit numbers. In a further cross-sectional study of 166 children (K-6) and 79 high ability children (Grades 3-6) it was found that the children's internal representations of numbers were highly imagistic, and that their imagistic configurations embodied structural development of the number system to widely varying extents, and often in unconventional ways (Thomas, Mulligan and Goldin, 1994). These studies raise the question of how children's representations of the number system are linked with their conceptual understanding of numeration and whether their representational capabilities influence the way they apply this knowledge in problem-solving situations.

The overall research aim is to describe in as much detail as possible children's internal representational capabilities when solving a wide range of tasks related to counting and numeration. This paper discusses one aspect of this research which involves an indepth, descriptive analysis of high ability children's internal representations of the number sequence 1-100. This analysis is based on Goldin's (1987a,b; 1988) model of problem-solving competency structures because our data consistently reflected aspects of internal representation previously described by Goldin. Goldin's model also contained features that are helpful in describing the development of children's conceptual understanding of numeration (Goldin and Herscovics, 1991). Further, we analyse the role of imagery in representation and in the construction of relational understanding in mathematics (Brown and Wheatley, 1989; Mason, 1992; Presmeg, 1986, 1992).

Goldin's model of competence in mathematical problem solving is based on the idea of cognitive representational systems internal to problem solvers, as distinct from (external) task variables and task structures (Goldin, 1988, 1992). We distinguish carefully between external representation (a structured environment with which the child is interacting that may include, for example, actual physical objects to manipulate and actions in response to that environment) and internal imagistic representation (a theoretical construct to describe the child's inner cognitive processing). We consider three of the five types of internal representational systems discussed by Goldin: (a) verbal/syntactic systems (using mathematical vocabulary, developing precision of language, self-reflective descriptions); (b) imagistic systems (non-verbal, non-notational representations, e.g. visual or kinaesthetic); and (c) formal notational systems (using notation, relating notation to conceptual understanding, creating new notations). These systems develop over time through three stages of construction: (i) inventive/semiotic, in which characters in a new system are first given meaning in relation to previously-constructed representations; (ii) structural development, where the new system is "driven" in its development by a previously existing system; and (iii) autonomous, where the new system of representation can function independently of its precursor.

Maher, Davis and Alston (1992) point out the complexity a teacher faces in trying to identify children's representations. They report on a study of the how children's representations are used and how they grow. It is suggested that the process of building up a mental image is an "extremely arduous task" (p. 260) for any learner. At times it involves shutting out most incoming signals and engaging in deep thought. Because of the value and fragility of a mental representation it is essential that the teacher recognises the imagery so that further development of understanding can occur. If a child's representation reflects developing structure, the teacher needs to build on it.

The Role of Imagery in Representation

The role of imagery in the representation of mathematical ideas has been described by a number of researchers (Bishop, 1989). For example Hershkowitz and Markovitz (1992) have emphasized the importance of visualization of mathematical concepts and development of advanced visual thinking. and Bobis (1993) has identifed the role of visualization in estimation of number. Recent work (Brown and Wheatley, 1989; Presmeg, 1986, 1992) in which individual students' thinking was probed in clinical interviews indicated that students use imagery in the construction of mathematical meaning. Brown and Presmeg (1993) assert that learning frequently involves the use of imagery and it must include very abstract and vague forms of imagery. Presmeg (1986) identified five types of visual imagery used by students: (i) concrete, pictorial imagery (pictures in the mind); (ii) pattern imagery (pure relationships depicted in a visual-spatial scheme); (iii) memory images of formulae; (iv) kinaesthetic imagery (involving muscular activity, e.g. fingers 'walking'; and (v) dynamic (moving) imagery involving the transformation of concrete visual images. Recent findings revealed wide differences in the types and facility of imagery used by students in problem solving. Students with a greater relational understanding of mathematics tended to use more abstract forms of imagery such as dynamic and pattern imagery while students with less relational understanding tended to rely on concrete and memory images (Brown and Presmeg, 1993). In this paper we draw upon Brown and Presmeg's notion of a dynamic image in the context of high ability children's imagistic representations of the number sequence.

Mason (1992) suggests that images can be viewed as either *eidetic*, fully formed from something that is presented, or *constructed*, built up from other images. The process of constructing meaning continues as the mental picture is described, drawn, compared and discussed. He asserts that it is not just the creation of a mental image but the use of that image as a mental 'screen' in visual thinking that constitutes mathematics learning. He questions the relationship between the relative accessibility of static and dynamic images and what makes some images enduring and others transitory. Mason further suggests that in order for students to access images they need to be active in processing the images, 'looking through' rather than 'looking at' the 'mental screen', regardless of the mode of external representation.

We extend Mason's notion to analyse children's active processing of internal images as static or dynamic in nature: static meaning a fixed representation, and dynamic as a representation that is changing and moving. We question whether the dynamic and static images are conventional or uniform in nature, and whether children have a range of available images. Further studies (Rubin and Russell, 1992) assert that people who are adept with number operations e.g. computing, comparing, and estimating, have a non-uniform view of the whole number system. We infer from children's external representations a static or dynamic internal representation of number and their view of the number system.

Methodology

Our study was designed to explore the relationship between children's counting, grouping and place value knowledge and their understanding of the structural development of the number system. We inferred the diverse concepts, structures and/or processes that occur inside children's heads from the observation, classification and measurement of the various kinds of behaviour displayed. Two samples of children were selected for comparative purposes, and administered task-based problem-solving interviews. A cross-sectional sample consisted of 166 children in Grades K-6, randomly chosen from 8 State schools in the Western Region of New South Wales (NSW). This sample represented a wide range of mathematical abilities. A high ability sample consisted of 92 children from Grades 3-6, assessed by teachers for participation in a Program for Gifted and Talented students from 75 country and city schools in NSW. The children were interviewed to ascertain their understanding of numeration using group administered and individual tasks. The children in the high ability sample were interviewed individually on selected numeraton and visualization tasks. The analysis discussed in this paper examines the visualization task where children were asked to close their eyes and to imagine the numbers from 1 to 100, then asked to draw the pictures that they saw in their minds. They were also asked to explain the image and their drawing. This task was asked prior to other numeration tasks so that responses could not be influenced by experiences with the other questions.

Analysis of External Representations

The interview transcripts and the pictorial and notational recordings of the counting sequence 1-100 task for the 92 high ability Grade 3-6 children were analysed. The external representations were categorised according to three dimensions: (i) the type of imagistic representation identified by the pictorial, ikonic and notational recordings; (ii) the level of creative structural development of the number system, and (iii) evidence of a static or dynamic nature of the image. In some cases these representations revealed idiosyncratic, highly individualistic images. The static or dynamic nature of the image was defined according to whether the recordings and the children's explanations of their representations described fixed or moving (or changing) entities.

In the cross-sectional sample 3% of the children displayed dynamic images of the number sequence and in the sample of high achieving children 18.5% had dynamic images. We discuss features of the dynamic examples of the representations of the high ability children. Evidence of highly structural imagistic internal representations for the children's developing numeration systems were found. The dynamic images were all notational, although one child saw the symbols in association with pictures. Examples of dynamic images included groups of numerals moving around, numerals going past in order, numerals appearing one at a time on a moving screen, sequences of numerals in line appearing and then being replaced by other sequences and whole arrays of numerals moving. Although most initial images did not involve array structures, when asked to think of the numbers in rows and columns many children drew pictures of arrays.

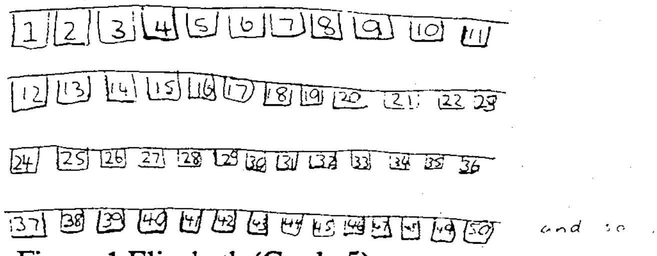


Figure 1 Elizabeth (Grade 5)

Elizabeth (Grade 5) recorded the number sequence in partial sequences of varying lengths (Figure 1). She explained that "each line goes up... one each time". Renee (Grade 5) recorded a standard array which was as a "board... moving to the right". Ben explained that he "saw ten rows of numbers each containing the set of ten numbers that come next in the sequence... the rows move along to be replaced by the next few."

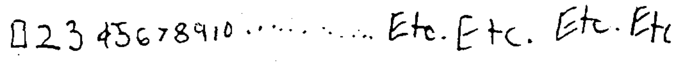


Figure 2 Adrian (Grade 3)

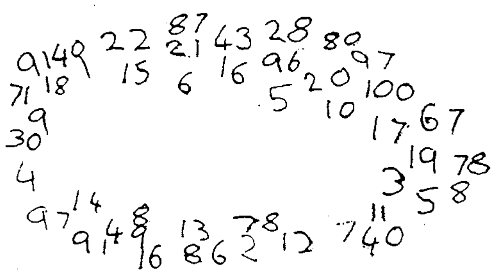


Figure 3 Caedyn (Grade 4)

Adrian (Figures 2) produced the sequence of numerals where each numeral moved past very quickly and Caedyn (Figure 3) produced the numerals in a swirl moving anti-clockwise. When both children were asked to again close their eyes and think of the numbers in rows and columns, Adrian said he "couldn't see them in rows because they were moving so fast" whereas Caedyn produced an array with the sequence going down in columns of ten. Adrian's dynamic visualisation was so dominant that he could not organize the numerals as directed whereas we infer that Caedyn's internal representation of the numeration system involved a structure of ten tens. Caedyn's imagery was more flexible and we can infer a higher level of structure in the number system available to her.

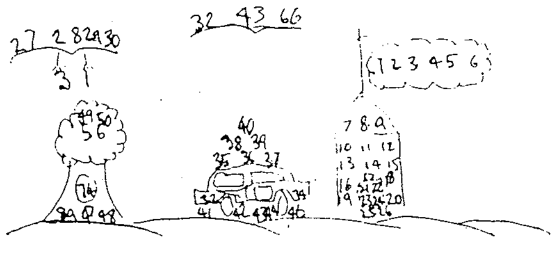
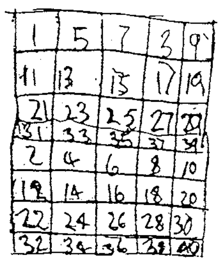


Figure 4 Clint (Grade 5)



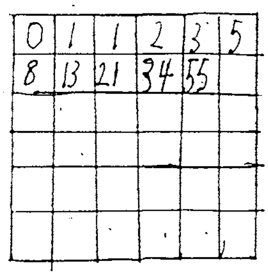


Figure 5 Clint (Grade 5) Figure 6 David (Grade 6)

Figures 4 shows Clint's spontaneous response to the visualization task where "the numbers were moving around like people" whereas Figure 5 shows his prompted response to the suggestion to think of the numbers in rows and columns. No evidence of structure was demonstrated in Clint's external imagery. His spontaneous response showed the linear sequence in a highly creative context and his prompted response demonstrated a relationship to conventional experiences with arrays but without any understanding of the significance to the number sequence. David (Grade 6) described a pictorial representation of groupings of objects, "...as I thought, everything suddenly became more numerous... like first there was one of everything, then there was two of everything and then there were three trees, suns, cats, dogs, people and it kept on going." Figure 6 shows the 6 by 6 grid with random numerals that David drew when prompted to think of numbers in rows and columns. David's spontaneous imagery focussed on the cardinal aspect of the numbers as they appeared in sequence and there was no evidence of structure given by the prompted response.

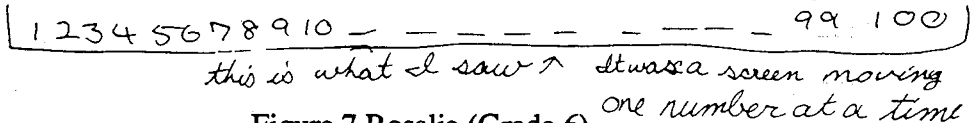


Figure 7 Rosalie (Grade 6)

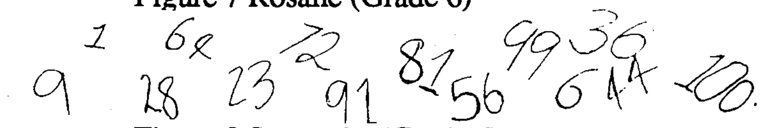


Figure 8 Samantha (Grade 6)

Rosalie (Grade 6) described "a screen moving one number at a time" and Samantha (Grade 6) described "the numbers 1 to 100 just floating around, all mixed up" (Figures 7 and 8). When asked to think of the numbers in rows and columns both children drew pictures of 10 by 10 grids with the numerals 1 to 100 in rows of ten. The spontaneous visualisations were dynamic but did not suggest advanced structural development. When prompted though, both girls produced conventional static array imagery for the number sequence. From this we infer that both are well on the way to developing an autonomous structure.

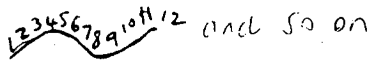


Figure 9 Colin (Grade 5)

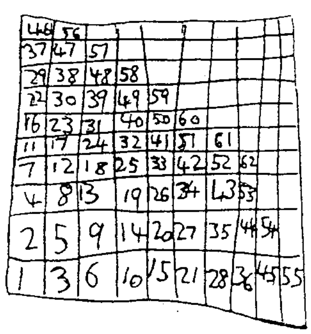


Figure 10 Colin (Grade 5)

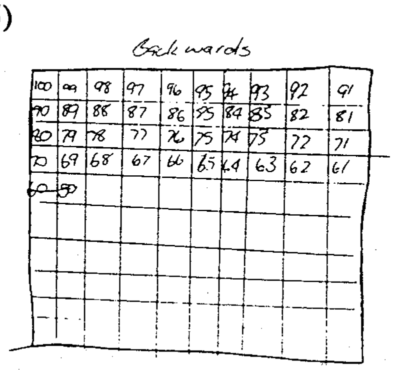


Figure 11 Christopher (Grade 5)

Colin (Grade 5) described the numbers moving along a wave like line (Figure 9) and Christopher (Grade 5) explained "I saw all the numbers from 1 to 100 beaming at me and lighting up like neon signs... the numbers did this in order then disappeared when I openned my eyes... they were moving around... then they would flash... once they had done that they disappeared." Figures 10 and 11 show the prompted responses from Colin and Christopher. Colin started at the bottom left corner of the grid and then filled in diagonals of increasing size moving up to the top right corner and Christopher displayed the numbers backwards from 100 to 1 in rows of ten. These boys showed high levels of acheivement on other numeration tasks and their prompted imagery involved 10 by 10 grids being 'filled-in' in non-conventional ways.

All examples of representations reported here were made by children identified by their teachers as high-acheivers in mathematics and with attitudes reflecting self-confidence in their abilities. Although some spontaneous dynamic imagery did not reveal structure, when children were prompted, varying aspects of structure appeared. These children showed evidence of having access to a variety of internal images and of developing relational understanding of the numeration system.

Conclusions and Limitations

We would ultimately like to be able to describe in detail children's internal representations of the numeration system, and how these representations develop. From the external representations produced by the high ability children, we have attempted to infer dynamic aspects of their internal representation, and from this in turn to infer some description of the structural development of the system that has taken place. We found in general a wider diversity of representations of the counting sequence 1-100 than might have been expected and a higher percentage of dynamic imagery than for the average/lower ability children. This diversity occurred at each grade level. We have described the children's spontaneous responses to the visualization task as presented to them. Those children who gave an image that did not involve an array as their response were then prompted to think of the numbers in rows and columns. Other responses might have occurred if the children had been prompted in other ways, for example, to imagine moving numbers, or to group the numbers in tens. Other representations may have been available to the children, with just one of several possible internal image configurations having been selected for recording or elaboration. Thus we are probably gaining but a partial description of each child's internal representational capabilities. Many questions are raised but unanswered. Why do some have the capability of spontaneously visualizing the counting sequence in a dynamic way? Can static external representations represent ("carry the meaning of") dynamic internal representations? Further research is needed to shed light on how children construct their personal numeration systems, and how they structure them over time. We hypothesize that the further-developed is the structure of a child's internal representational system for the counting numbers (e.g., kinaesthetic, auditory, or visual/spatial representation of the counting sequence that embodies grouping-by-tens), the more coherent and well-organized will be the child's externally-produced representations, and the wider will be its range of numerical understandings. We aim to investigate further whether children who have access to a variety of imagery to represent their internal structures will be more capable of developing relational understanding.

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